Sri Chaitanya Educational Institutions | Infinity Learn

39th Indian National Mathematical Olympiad – 2025

Time: 4.5 hours January 19, 2025

Instructions:

- Calculators (in any form) and protractors are not allowed. Rulers and compasses are allowed.
- All questions carry equal marks. Maximum marks: 102.
- No marks will be awarded for stating an answer without justification.
- Answer to each question should start on a new page. Clearly indicate the question number.

1. Consider the sequence defined by $a_1 = 2, a_2 = 3$, and

$a_{2k+2} = 2 + a_k + a_{k+1}$ and $a_{2k+1} = 2 + 2a_k$

for all integers $k \geqslant 1$. Determine all positive integers n such that $\frac{a_n}{n}$ is an integer.

Sol. Only such n are $\left(2^k - 1\right) \forall k \in N$.

Claim-I: - a_{2k} is odd & a_{2k+1} is even

Note $a_3 = 2 + 4 = 6.$

$a_4 = 2 + a_1 + a_2 = 2 + 2 + 3 = 7$

$a_5 = 2 + 2a_2 = 8$

$a_6 = 2 + a_2 + a_3. = 2 + 3 + 6 = 11$

$a_7 = 2 + 2a_3 = 14$

$a_8 = 2 + a_3 + a_4 = 2 + 6 + 7 = 15$

$a_{15} = 2\left(1 + a_7\right) = 30$

We can prove the result by induction.

Let it be true for all a_n for $n \leqslant 2k$

Now. $a_{2k+1} = 2\left(1 + a_k\right) =$ even.

$a_{2k+1} = 2\left(1 + a_k\right) =$ even.

$a_{2k+2} = 2 + \underbrace{a_k + a_{k+1}}$ One of them is odd & other is even

= odd

Claim-II: $a_{2^n - 1} = 2\left(2^n - 1\right).$

We note $a_1 = 1, a_3 = 6, a_7 = 14, a_{15} = 30$

Assume result for $a_{2^n - 1}$ is $a_{2^n - 1} = 2\left(2^n - 1\right) = 2^{n+1} - 2$

Now $a_{2^{n+1} - 1} = 2\left(1 + a_{2^n - 1}\right) = 2\left(1 + 2^{n+1} - 2\right)$

$= 2\left(2^{n+1} - 1\right)$

Now we need to prove that $n \nmid a_n$ for $n \neq 2^k - 1$.

For this by induction we can prove $a_n > n. \quad \forall n \in N$.

<u>Claim-III</u>:- $a_n \leqslant 2n$ with Equality iff $n = 2^k - 1$

i.e $\begin{cases} a_n < 2n & \text{if} \quad n \neq 2^k - 1 \\ a_n = 2n & \text{if} \quad n = 2^k - 1 \end{cases}$

Clearly, we have base cases a_1 to a_8.

So we assume result for true till $n \leqslant 2m$.

For $a_{2m+1} = 2(1 + a_m)$

If $m = 2^k - 1 \Rightarrow 2m + 1 = 2^{k+1} - 1$.

$$a_{2m+1} = a_{2^{k+1}-1} \qquad = 2\left(1 + 2\left(2^k - 1\right)\right)$$

$$= 2\left(2^{k+1} - 1\right)$$

If $m \neq 2^{k-1} \Rightarrow a_m < 2m$.

$a_{2m+1} < 2(1 + 2m)$

For $a_{2m+2} = 2 + a_m + a_{m+1}$.

Clearly one of m or m + 1 is not of form $\left(2^k - 1\right)$

$\therefore$ ex. $a_m < 2m$ or $a_{m+1} < 2m + 1$

$\Rightarrow a_{2m+2} < 2 + 2m + 2m + 1 = 2(2m + 2)$.

2. Let $n \geq 2$ be a positive integer. The integers $1, 2, \cdots, n$ are written on a board. In a move, Alice can pick two integers written on the board $a \neq b$ such that $a + b$ is an even number, erase both a and b from the board and write the number $\frac{a+b}{2}$ on the board instead. Find all n for which Alice can make a sequence of moves so that she ends up with only one number remaining on the board.

Note. When $n = 3$, Alice changes $(1, 2, 3)$ to $(2, 2)$ and can't make any further moves.

Sol.

Let no n be good if she ends up with. only one integer. answer only non-good numbers are 2,3,4 & 6.

So 1,5 and $n \geqslant 7$ are all good.

Claim-I for odd $n \geqslant 7$ the last no left can be (n-3).

It's easy to check, this for $n = 7 < 9$.

1 2 3 4 5 6 7

1 2 3 4 6 6

1 2 3 5 6

2 2 5 6

2 4 5

3 5

4

Claim II: If odd n is good so is (n+4)

For $n \geq 9$. Consider. n + 4 numbers.

1, 2, ... n, (n+1), (n+2), (n+3), (n+4

(n-3),(n+1), (n+2), (n+3), (n+4)

(n-3), (n+2), (n+2), (n+4)

(n-3), (n+3) (n+2)

n (n+2)

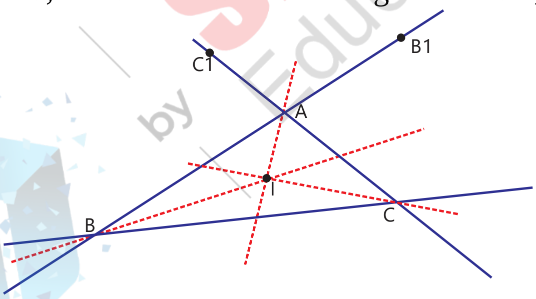
(n+1)

3. Euclid has a tool called splitter which can only do the following two types of operations:

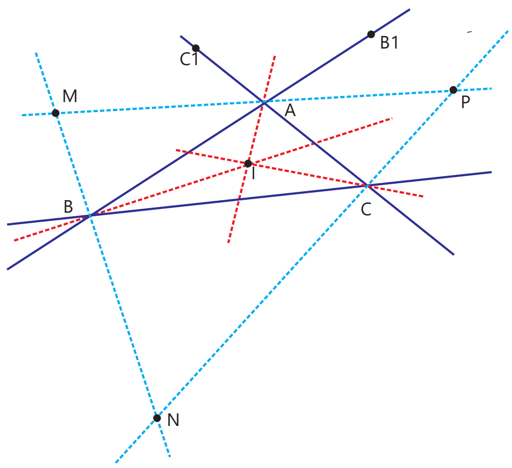
- Given three non-collinear marked points X, Y, Z, it can draw the line which forms the interior angle bisector of $\angle XYZ$.
- It can mark the intersection point of two previously drawn non-parallel lines.

Suppose Euclid is only given three non-collinear marked points A, B, C in the plane. Prove that Euclid can use the splitter several times to draw the centre of the circle passing through A, 13, and C.

Sol. **Step-I:** Join AB, BC, CA and draw interior angle bisectors, concurrent at I.



Step-II: Consider points C_1 & B_1 on CA & BA entered base and draw interior angle bisector of $\angle C_1AB_1$ (which is exterior angle bisector of $\angle A$). Likewise for $\angle B$ & $\angle C$ Let these intersect at M, N, T.



Step-III: Mark mid points of MT as X and IN as Y Observe X & Y are extremities of diameter of circumcircle of ΔABC
Last step Mark midpoint of XY as circumcircle of ΔABC.

4. Let $n\geqslant 3$ be a positive integer. Find the largest real number t_n as a function of n such that the inequality

$$\max\left(|a_1+a_2|,|a_2+a_3|,\ldots,|a_{n-1}+a_n|,|a_n+a_1|\right)\geqslant t_n\cdot\max\left(|a_1|,|a_2|,\cdots,|a_n|\right)$$

holds for all real numbers $a_1,a_2,\cdots,a_n$.

Sol.

Claim-I: If n is even then $t_n=0$.

Consider $a_{2k+1}=1$ & $a_{2k}=-1$.

$$\Rightarrow 0\geqslant t_n\cdot\max\left(|a_1|,\ldots|a_n|\right)$$

$$\Rightarrow \quad t_n=0.$$

Claim-II: - For odd n, $t_n=\frac{2}{n}$.

let $M=\max\{|a_1+a_2|,|a_2+a_3|\cdots(a_n+a_1)\}$

WLG $|a_1|=\max\{|a_k|\}$, Let $a_1=x$.

$|a_1+a_2|\leqslant M \quad \Rightarrow -M\leqslant x+a_2\leqslant M$

$\quad -x-M\leqslant a_2\leqslant M-x$

$|a_2+a_3|\leqslant M \quad \Rightarrow -M\leqslant a_3+a_2\leqslant M.$

$\quad -M+x\leqslant -a_2\leqslant x+M$

$\quad -2M+x\leqslant a_3\leqslant 2M+x$

$|a_3+a_4|\leqslant M \quad \Rightarrow -2M-x,\leq a_3\leqslant 2M-x$

$\quad -M\leqslant a_4+a_3\leqslant M$

$\quad -3M-x\leqslant a_4\leqslant 3M-x$

By induct we can say

$-(n-1)M-x\leqslant a_n\leqslant (n-1)M-x$ for n even.

$-(n-1)M+x\leqslant a_n\leqslant (n-1)M+x$ for n odd,

As $a_1=2$ As n is odd.

$$\text{Also}\quad \left.\begin{array}{ll} -(n-1)M+2x & \leqslant a_n+a_1\leqslant (n-1)M+2x. \\ -M & \leqslant a_n+a_1\leqslant M. \end{array}\right\}$$

For this 2 to be ture

$-(n-1)M+2x\leqslant M.$ & $-M\leqslant (n-1)M+2x$

$2x\leqslant nM.$ & $-nM\leqslant 2x$

$$\Rightarrow \quad -nM\leqslant 2x\leqslant nM$$

$$\therefore \quad |x|\leqslant\frac{nM}{2}.$$

$$t_n\,|x|\leqslant M.$$

If $\exists$ some $t_n > \frac{2}{n}$ then w can take a_i' ' as above to get $t_n |x| > M$.

$$\Rightarrow t_n \leqslant \frac{2}{n}.$$

Construction:- Let $n = 2k+1$

Take $|a_1| = x \quad |a_2| = 3x \cdots \quad |\cdot a_{k+1}| = (2k+1)x = nx$.

$|a_{k+2}| = (2k-1)x, \quad |a_{k+3}| = (2k-3)x \ldots |a_n| = x.$

With each adjacent a_i with opposite signs.

ie, $a_k a_{k+1} < 0$

$a_1 = x, \quad a_2 = -3x, \quad a_3 = 5x, \ldots a_k = \pm(2k-1)x$

$a_{k+1} = \pm(2k+1)x, \quad a_{k+2} = \pm(2k-1)x, \quad \cdots \quad a_n = x.$

$\therefore$ Each $|a_k + a_{k+1}| = 2x$ & t_n Max $\{|a_i|\} = \frac{2}{n}$nx=2x.

5. Greedy goblin Griphook has a regular 2000-gon, whose every vertex has a single coin. In a move, he chooses a vertex, removes one coin each from the two adjacent vertices, and adds one coin to the chosen vertex, keeping the remaining coin for himself. He can only make such a move if both adjacent vertices have at least one coin. Griphook stops only when he cannot make any more moves. What is the maximum and minimum number of coins that he could have collected?

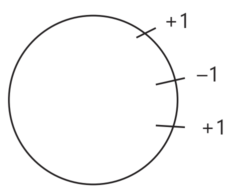
Sol. **<u>Claim</u>**: - The maximum no f coins collected = 1998.

Proof:- Let us assign weights to vertices alternatively $(+1)$ & (-1)

Initially sum of all weights = 0.

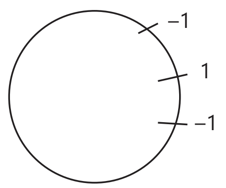
Consider an operation on vertex (-1)

New Sum $= 0 - (2) + (-1)$
$= -3$
$= 0(\bmod 3).$



Consider an operation on vertex (1)

New sum $= 0 - (-2) + (1)$
$= 3 \equiv 0 \ (\bmod 3)$



So Sum (mod 3) remains invariant under operations.

Support if max coins he gets is 1999, then the final sum would be $\neq 1$

$\Rightarrow$ Contradiction.

Hence max no of coins $\leqslant 1998$.

We give configuration for 1991 coins.

We operate on following triples.

$(1,2,3),(2,3,4),(3,4,5)\ldots(1998,1999,2000)$

<u>Claim-II:</u> - Minimum no of coins he can get = 400. As in any 5 consecutive coins he can do at least one operation hence min no of coins he can get $= \left[\frac{2000}{5}\right] = 400$

6. Let $b \geqslant 2$ be a positive integer. Anu has an infinite collection of notes with exactly $b-1$ copies of a note worth $b^k - 1$ rupees, for every integer $k \geqslant 1$. A positive integer n is called payable if Anu can pay exactly n^2+1 rupees by using some collection of her notes. Prove that if there is a payable number, there are infinitely many payable numbers.

Sol.

If b = 2 then Anu has infinite collection of 1 note of denominations $b^k - 1$, which means 1 note of $2^k - 1$ which can be $1, 3, 7, 15 \ldots \ldots$.

If b = 3 then Anu has infinite collection of 2 notes of denominations $3^k - 1$; which means 2 notes of denominations. $2, 8, 26, 80 \ldots \ldots$. and so on so forth

Now let us assume for $n = \lambda; \lambda^2 + 1$ can be payable with certain collection of notes

$\Rightarrow \lambda^2 + 1 = \sum_{i=1}^{\infty} p_i \left(b^i - 1\right)$ where $p_i \in \{0, 1, 2, \ldots b-1\}$

Consider $y = b\left(\lambda^2 + \lambda + 1\right)$ then $y^2 + 1 = \left\{\left(\lambda^2+1\right)^2 + 2\lambda\left(\lambda^2+1\right) + \lambda^2 + 1\right\} b^2$

$$= \left(\lambda^2+1\right)\left(\lambda^2 + 2\lambda + 1\right) b^2$$

$$= \left(\lambda^2+1\right)(\lambda+1)^2 b^2$$

Hence if These is a payable number there are infinitely many payable numbers@ b varies.